Stochastic Process

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Stochastic Process
(Random Process)

- A stochastic process $\{X(t): t \in T\}$ is a family of random variables indexed by a parameter t, which runs over an index set T.
- The parameter t usually denotes time
- For any specific time t, $X(t)$ is a random variable.
- The index set T is called the parameter set.
- If T is countable, $\{X(t)\}$ is discrete time stochastic process. If T is an interval, finite or infinite, $\{X(t)\}$ is said to be continuous time stochastic process.
- The set of possible values of $X(t)$ at any time t is called the state space, S.

Stochastic Process
(Random Process)

Example: Suppose that, a business office has five telephone lines and that any number of these lines may be in use at any time, the telephone lines are observed at regular interval of 2 minutes.

X: Number of lines in use in every 2 minutes

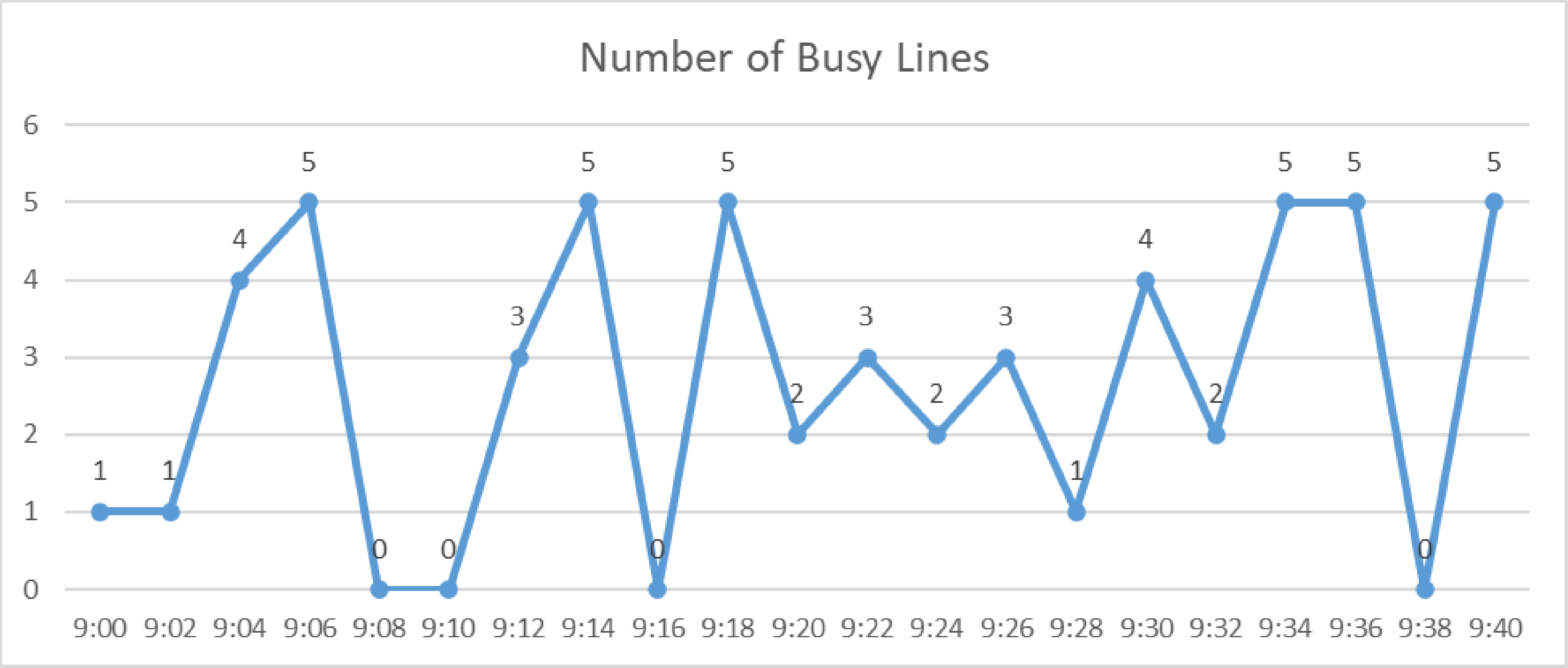
Then for T=0, 1, 2, 3, … ,

X(t): {X(0), X(1), X(2), … }   e.g. X(t): {3, 2, 5, 0, 1, 0, 3, … }

Here, {X(t): t ϵ T} is a stochastic process with parameter space, T= {0, 1, 2, 3, … } and State space, S= {X: 0, 1, 2, 3, 4, 5}

Stochastic Process
(Random Process)

An observation of this process might be one like below-



Stochastic Process
(Random Process)

Examples of some stochastic processes:

1. Random walk model:

Let, X_t is a random variable that can be any of the two states [up (+1) or down (-1)] at time t.

X_t	+1	-1
probability	p	1-p

Then, the process $R_t = R_{t\_1} + X_t$ is called a random walk model

2. Counting process

A stochastic process $\{N(t); t > 0\}$ is a counting process if N(t) represents the total number of events that have occurred in time t.

3. Birth & death process

A stochastic process $\{N(t); t \geq 0\}$ with states $\{n = 0,1,2, \dots\}$ for which transition from state n may go only to either of the states (n-1) or (n+1) is a birth and death process.

Autocorrelation
(serial correlation)

- In statistics, the autocorrelation of a random process describes the correlation between values of the process at different times, as a function of the two times or of the time lag.
- Let X be some repeatable process, and i be some point in time after the start of that process. Then X_i is the value (or realization) produced by a given run of the process at time i. Suppose that the process is further known to have defined values for mean μ_i and variance σ_i^2 for all times i. Then the definition of the autocorrelation between times s and t is-

$$R(s,t) = \frac{E[(X_t - \mu_t)(X_s - \mu_s)]}{\sigma_t \sigma_s} = Corr(X_t, X_s)$$

Markov Process

- Consider a finite or countably infinite set of points (t_0,, t_1, … , t_n, t), where $t_0<t_1<\ldots<t_n<t$ and t, $t_i \in T$ (i= 0, 1, 2, … , n); T is a parameter space of the process {x(t)}.
- The dependence exhibited by the process {X(t): t ϵ T} is called Markovian dependence if the conditional distribution of X(t) for a given value of $X(t_1)$, $X(t_2)$, … $X(t_n)$ depends only on $X(t_n)$, which is the most recent known value of the process, i.e., if

$$P[X(t) = x|\, x(t_n) = x_n, X(t_{n-1}) = x_{n-1}, \ldots, X(t_0) = x_0] = P[X(t) = x|\, x(t_n) = x_n]$$

The stochastic process exhibiting the property (Markov property) is called a Markov process.

Markov Process

Different types of Markov process		
	State space	
Parameter space	Discrete	Continuous
Discrete	Markov Chain	Discrete parameter, continuous MP
continuous	Continuous parameter, discrete MC	Continuous parameter, continuous MP

Notations:

$p_i = Probability\ that\ the\ process\ is\ in\ state\ i$

$p_{ij}^{(n)} = n - step\ transition\ probability\ of\ state\ j\ form\ state\ i$

$= \ Probability\ that\ the\ process\ will\ move\ from\ state\ i\ to\ state\ j\ in\ n - steps$

Markov Process

Transition Probability Matrix (TPM):

A matrix containing probabilities of transition from one state to another. If there are k finite states of a Markov process, i.e. S={1, 2, … k}, then one-step transition probability matrix-

$$P = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ k \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & \cdots & k \end{array} \\ \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1k} \\ P_{21} & P_{22} & \cdots & P_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ P_{k1} & P_{k2} & \cdots & P_{kk} \end{bmatrix} \end{array}$$

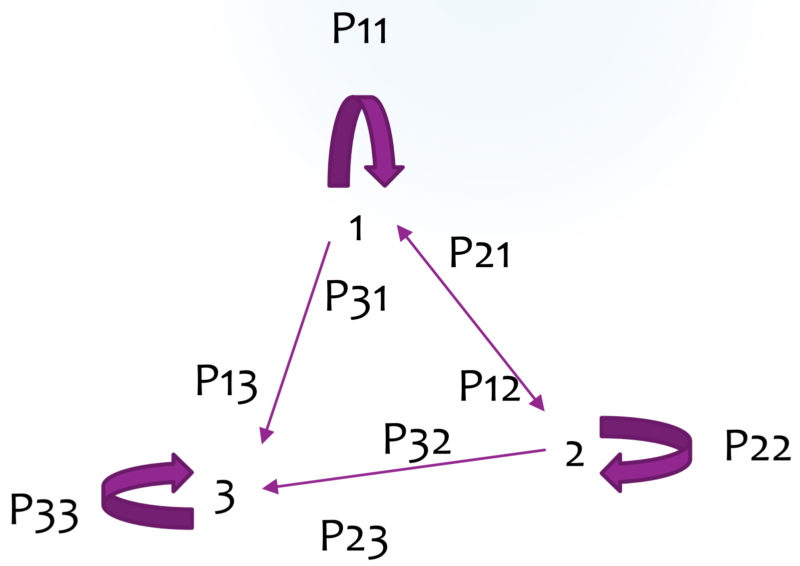
Such that, $i)\ P_{ij} \geq 0, \forall i, j \ \in S; ii)\ \sum_j P_{ij} = 1\ (row\ total\ 1)$

Markov Process (Example)

Example 1:

X= Living status. State space, S= {1, 2, 3}; where, 1= Healthy, 2= Sick, 3 = Dead.

$$P = \begin{matrix} \\ 1 \\ 2 \\ 3 \end{matrix} \begin{matrix} 1 & 2 & 3 \\ \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \end{matrix}$$



Here, $P_{11} > 0, P_{12} > 0, P_{13} > 0, P_{21} > 0, P_{22} > 0, P_{23} > 0,$
$P_{31} = 0, P_{32} = 0, P_{33} = 1$

Markov Process

Time-homogeneous Markov chains (or stationary Markov chains) are processes where

$$P[X_{t+1} = x|\, X_t = y] = P[X_t = x|\, X_{t-1} = y]$$

for all t. The probability of the transition is independent of t.

Markov Process

<u>Classification of states:</u>

- *Accessible state:* A state j is said to be accessible from a state i (written i → j) if a system started in state i has a non-zero probability $(P_{ij} > 0)$ of transitioning into state j at some point.
- *Communicating states:* A state i is said to communicate with state j (written i ↔ j) if both i → j and j → i.
- *Absorbing state:* A state i is called absorbing if it is impossible to leave this state. Therefore, the state i is absorbing if and only if $p_{ii} = 1 \; and \; p_{ij} = 0 \; for \; i \neq j$

Markov Process

Classification of states:

- *Transient state & recurrent state:* A state i is said to be transient if, given that we start in state i, there is a non-zero probability that we will never return to I (the process may not return to state i). State i is recurrent (or persistent) if it is not transient. Recurrent states are guaranteed (with probability 1) to have a finite hitting time.
- *Periodic state & aperiodic state:* A state i has period k if any return to state i must occur in multiples of k time steps. A state is said to be aperiodic if returns to state i can occur at irregular times (k=1).

Markov Process
(Example)

Example 2: There are two telephone lines in a office and any number of these lines may be in use at any given time. During a certain point of time, telephone lines are observed at regular interval of 2 minutes. The initial probability vector of the states is-

$$A_{1\times K} = (0.3, 0.5, 0.2)$$

And one-step transition probability matrix is-

$$P_{K\times K} = \begin{bmatrix} 0.2 & 0.6 & 0.2 \\ 0.3 & 0.5 & 0.2 \\ 0.1 & 0.4 & 0.5 \end{bmatrix}$$

Assuming, a homogenous Markov chain, determine that probabilities that no line, 1 line and 2 lines are being used at each of the following times: i) t=2, ii) t=3. (Assuming starting time t=0).

Markov Process
(Example)

Let, 0: No line is in use

1: 1 line is in use

2: 2 lines is in use.

Therefore, state space, S={0, 1, 2}

Given that, the initial probability vector of the states is-

$$P_{1\times K} = (0.3, 0.5, 0.2)$$

And one-step transition probability matrix is-

$$P_{K\times K} = \begin{bmatrix} 0.2 & 0.6 & 0.2 \\ 0.3 & 0.5 & 0.2 \\ 0.1 & 0.4 & 0.5 \end{bmatrix}$$

Markov Process
(Example)

i) For t=2:

$$P = P_{1\times K}P_{K\times K} = [0.3, 0.5, 0.2]\begin{bmatrix}0.2 & 0.6 & 0.2\\ 0.3 & 0.5 & 0.2\\ 0.1 & 0.4 & 0.5\end{bmatrix}$$
$$= [0.23, 0.51, 0.26]$$

i) For t=3:

$$P = P_{1\times K}P^2_{K\times K} = P_{1\times K}P_{K\times K}P_{K\times K} = [0.23, 0.51, 0.26]\begin{bmatrix}0.2 & 0.6 & 0.2\\ 0.3 & 0.5 & 0.2\\ 0.1 & 0.4 & 0.5\end{bmatrix}$$
$$= [0.225, 0.497, 0.278]$$

Markov Process

- <u>Steady State Probabilities:</u>

If for a irreducible (only one class, so that all states communicate) Markov Chain, all of the states are aperiodic and positive recurrent (it will return in a finite time), the distribution

$$A^{(n)} = A.P^n$$

converges as $n \to \infty$, and the limiting distribution is independent of the initial probabilities, A. In general,

$$\lim_{n\to\infty} p_{ij}^{(n)} = \lim_{n\to\infty} a_j^{(n)} = p_j$$

Markov Process

- <u>Steady State Probabilities:</u>

Furthermore, the values p_j are independent of i. These probabilities are called steady state probabilities. These steady state probabilities p_j satisfy the following state equations-

$$p_j > 0, \dots\dots\dots\dots (1)$$

$$\sum_{j=0}^{m} p_j = 1\,, \dots\dots\dots\dots (2)$$

$$p_j = \sum_{i=0}^{m} p_i p_{ij}\,, \qquad j = 0,1,2,\dots, m, \dots\dots\dots\dots (3)$$

Since there are m+2 equation in (2) & (3), and since there are m+1 unknowns, one of the equations is redundant. Therefore we will use m of the m+1 equations in equation (3) with equation (2).

Markov Process
(Example)

Example 3:

Find steady state probabilities for the Markov chain described in example 2.

Here, for steady state,

$$(p_0 \quad p_1 \quad p_2) = (p_0 \quad p_1 \quad p_2)\begin{pmatrix} 0.2 & 0.6 & 0.2 \\ 0.3 & 0.5 & 0.2 \\ 0.1 & 0.4 & 0.5 \end{pmatrix}$$

$$\Rightarrow (p_0 \quad p_1 \quad p_2) = (0.2p_0 + 0.3p_1 + 0.1p_2 \quad 0.6p_0 + 0.5p_1 + 0.4p_2 \quad 0.2p_0 + 0.2p_1 + 0.5p_2)$$

$$\Rightarrow \begin{aligned} p_0 &= 0.2p_0 + 0.3p_1 + 0.1p_2 \\ p_1 &= 0.6p_0 + 0.5p_1 + 0.4p_2 \\ p_2 &= 0.2p_0 + 0.2p_1 + 0.5p_2 \end{aligned}$$

Also, $p_0 + p_1 + p_2 = 1$

Markov Process
(Example)

Rewriting the above system of equations-

$$\Rightarrow \begin{array}{c} -0.8p_0 + 0.3p_1 + 0.1p_2 = 0 \dots\dots\dots (1) \\ 0.6p_0 - 0.5p_1 + 0.4p_2 = 0 \dots\dots\dots (2) \\ 0.2p_0 + 0.2p_1 - 0.5p_2 = 0 \dots\dots\dots (3) \\ p_0 + p_1 + p_2 = 1 \dots\dots\dots\dots (4) \end{array}$$

Using the equations (1), (2) & (4) we will find the steady state probabilities. First reducing the system by eliminating p_3.

$Eq.(1) + (-0.1) \times Eq.(4) \Rightarrow$

$$\begin{array}{l} -0.8p_0 + 0.3p_1 + 0.1p_2 = 0 \\ -0.1p_0 - 0.1p_1 - 0.1p_2 = -0.1 \\ \hline -0.9p_0 + 0.2p_1 \qquad\qquad = -0.1 \dots\dots (5) \end{array}$$

Markov Process
(Example)

$Eq.(2) + (-0.4) \times Eq.(4) \Rightarrow$

$$
\begin{array}{l}
0.6p_0 - 0.5p_1 + 0.4p_2 = 0 \\
-0.4p_0 - 0.4p_1 - 0.4p_2 = -0.4 \\
\hline
0.2p_0 - 0.9p_1 \qquad = -0.4 \ldots\ldots(6)
\end{array}
$$

$Eq.(5) + \frac{2}{9} \times Eq.(6) \Rightarrow$

$$
\begin{array}{l}
-0.9p_0 + 0.2p_1 = -0.1 \\
\frac{0.4}{9}p_0 - 0.2p_1 = -\frac{0.8}{9} \\
\hline
-0.9p_0 + \frac{0.4}{9}p_0 \qquad = -0.1 - \frac{0.8}{9} \\
\Rightarrow 0.77p_0 = 0.11 \Rightarrow p_0 = \frac{17}{77}
\end{array}
$$

Markov Process
(Example)

$From\ Eq. (5),$

$$-0.9p_0 + 0.2p_1 = -0.1$$
$$\Rightarrow p_1 = -\frac{1}{2} + \frac{9}{2}p_0 = \frac{38}{77}$$

$From\ Eq. (4),$

$$p_0 + p_1 + p_2 = 1$$
$$\Rightarrow p_2 = 1 - \frac{55}{77} = \frac{22}{77}$$

So, the steady state probabilities for the above stated Markov chain are- $p_0 = \frac{17}{77}$, $p_1 = \frac{38}{77}$, $p_2 = \frac{22}{77}$